DYNNIKOV COORDINATES ON VIRTUAL BRAID GROUPS

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ABSTRACT

We define Dynnikov coordinates on virtual braid groups. We prove that they are faithful invariants of virtual 2-braids, and present evidence that they are also very powerful invariants for general virtual braids.

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1. Virtual braid groups

The group of virtual braids VB_n, $n \geqslant 2$, on n strings was introduced by Kauffman [4] as a generalization of the classical braid group B_n. The most useful system of generators and defining relations of VB_n was introduced by Vershinin in [6]. The generators of VB_n are

$$\sigma_1, \ldots, \sigma_{n-1}, \rho_1, \ldots, \rho_{n-1}, \tag{1.1}$$

and the defining relations are the following:

$$\sigma_i\, \sigma_{i+1}\, \sigma_i = \sigma_{i+1}\, \sigma_i\, \sigma_{i+1}, \qquad \sigma_i\, \sigma_j = \sigma_j\, \sigma_i \quad \text{if} \quad |i-j| > 1, \tag{1.2}$$

$$\rho_i^2 = 1, i = 1, \ldots, n-1, \quad \rho_i\, \rho_{i+1}\, \rho_i = \rho_{i+1}\, \rho_i\, \rho_{i+1}, \quad \rho_i\, \rho_j = \rho_j\, \rho_i \quad \text{if} \quad |i-j| > 1, \tag{1.3}$$

$$\sigma_i \rho_j = \rho_j \sigma_i \quad \text{if} \quad |i-j| > 1, \qquad \rho_i\, \rho_{i+1}\, \sigma_i = \sigma_{i+1}\, \rho_i\, \rho_{i+1}. \tag{1.4}$$

Thus, the group generated by $\sigma_1, \ldots, \sigma_{n-1}$ with relations (1.2) is the braid group B_n; the group generated by $\rho_1, \ldots, \rho_{n-1}$ with relations (1.3) is the symmetric

group S_n; the relations (1.4) will be referred to as *mixed relations*. The last presented relation is equivalent to

$$\rho_{i+1}\,\rho_i\,\sigma_{i+1} = \sigma_i\,\rho_{i+1}\,\rho_i. \tag{1.5}$$

We remark that the relations

$$\rho_i\,\sigma_{i+1}\,\sigma_i = \sigma_{i+1}\,\sigma_i\,\rho_{i+1}, \qquad \rho_{i+1}\,\sigma_i\,\sigma_{i+1} = \sigma_{i+1}\,\sigma_i\,\rho_i \tag{1.6}$$

do not hold in VB_n, and these relations will be referred to as *forbidden* relations. (Adding these forbidden relations yields the so-called *braid permutation group* [3]).

There is a natural epimorphism $\pi : VB_n \to S_n$ defined by

$$\pi(\sigma_i) = \pi(\rho_i) = \rho_i, \quad i = 1, \ldots, n-1.$$

The kernel $\ker(\pi)$ is called the *virtual pure braid group* and is denoted by VP_n. Generators and relations for VP_n are described in [1]. It is easy to see that VB_n is a semidirect product: $VB_n = VP_n \rtimes S_n$.

2. Coordinates on braid groups

In [2, Ch. 8], an action of the braid group B_n on the integer lattice $\mathbb{Z}^{2n}$ by piecewise-linear bijections is defined. For the reader's convenience we recall the definition. For $x \in \mathbb{Z}$ denote $x^+ = \max\{0, x\}$ and $x^- = \min\{x, 0\}$. Define actions

$$\sigma,\, \sigma^{-1} : \mathbb{Z}^4 \to \mathbb{Z}^4$$

on $(a, b, c, d) \in \mathbb{Z}^4$ as follows:

$$(a, b, c, d) \cdot \sigma = (a + b^+ + (d^+ - e)^+,\ d - e^+,\ c + d^- + (b^- + e)^-,\ b + e^+), \tag{2.1}$$

$$(a, b, c, d) \cdot \sigma^{-1} = (a - b^+ - (d^+ + f)^+,\ d + f^-,\ c - d^- - (b^- - f)^-,\ b - f^-), \tag{2.2}$$

where

$$e = a - b^- - c + d^+, \qquad f = a + b^- - c - d^+. \tag{2.3}$$

For a given vector $(a_1, b_1, \ldots, a_n, b_n) \in \mathbb{Z}^{2n}$ we define the action by $\sigma_i^{\varepsilon} \in B_n$, where $i = 1, \ldots, n-1$:

$$(a_1, b_1, \ldots, a_n, b_n) \cdot \sigma_i^{\varepsilon} = (a'_1, b'_1, \ldots, a'_n, b'_n), \tag{2.4}$$

where $a'_k = a_k$, $b'_k = b_k$ if $k \neq i, i+1$, and

$$(a'_i, b'_i, a'_{i+1}, b'_{i+1}) = \begin{cases} (a_i, b_i, a_{i+1}, b_{i+1}) \cdot \sigma, & \text{if } \varepsilon = 1, \\ (a_i, b_i, a_{i+1}, b_{i+1}) \cdot \sigma^{-1}, & \text{if } \varepsilon = -1. \end{cases} \tag{2.5}$$

For a word w in the alphabet $\{\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}\}$ we define an action by w:

$$(a_1, b_1, \ldots a_n, b_n) \cdot w = \begin{cases} (a_1, b_1, \ldots a_n, b_n), & \text{if } w = 1, \\ ((a_1, b_1, \ldots a_n, b_n) \cdot \sigma_i^{\varepsilon}) \cdot w', & \text{if } w = \sigma_i^{\varepsilon} w'. \end{cases} \tag{2.6}$$

It can be shown that the above action by B_n on $\mathbb{Z}^{2n}$ is well defined, i.e. if two words w_1 and w_2 present the same element of the braid group B_n then

$$(a_1, b_1, \dots, a_n, b_n) \cdot w_1 = (a_1, b_1, \dots, a_n, b_n) \cdot w_2$$

for any vector $(a_1, b_1, \dots, a_n, b_n) \in \mathbb{Z}^{2n}$. By the *Dynnikov coordinates* of a braid we will mean the vector $(0, 1, \dots, 0, 1) \cdot w$, where w is a word representing that braid.

Example 2.1. Actions by some elements of B_2 on $(0, 1, 0, 1) \in \mathbb{Z}^2$ are as follows:

$$\begin{array}{ll} (0,1,0,1)\cdot\sigma_1 = (1,0,0,2), & (0,1,0,1)\cdot\sigma_1^{-1} = (-1,0,0,2), \\ (0,1,0,1)\cdot\sigma_1^2 = (1,-1,0,3), & (0,1,0,1)\cdot\sigma_1^{-2} = (-1,-1,0,3), \\ \vdots & \vdots \\ (0,1,0,1)\cdot\sigma_1^k = (1,-k+1,0,k+1), & (0,1,0,1)\cdot\sigma_1^{-k} = (-1,-k+1,0,k+1), \end{array}$$

where $k \in \mathbb{N}$. Also, acting by some elements of B_3 on $(0, 1, 0, 1, 0, 1) \in \mathbb{Z}^6$ we have

$$\begin{aligned} (0,1,0,1,0,1)\cdot\sigma_1\sigma_2^{-1} &= (1,0,-2,0,0,3), \\ (0,1,0,1,0,1)\cdot\sigma_1\sigma_2\sigma_1 &= (2,0,1,0,0,3). \end{aligned}$$

Remark 2.2. It is also shown in [2] that Dynnikov coordinates are faithful invariants of braids, i.e. if $(0, 1, \dots, 0, 1) \cdot w_1 = (0, 1, \dots, 0, 1) \cdot w_2$ then $w_1 = w_2$ in B_n; thus, Dynnikov coordinates are very useful for solving the words problem in B_n.

Here is an outline of the proof: there is a bijection between vectors in $\mathbb{Z}^{2n}$ and integer laminations of a sphere with $n + 3$ punctures $P_0, P_1, \dots, P_{n+1}, P_\infty$; under this bijection, our action of B_n on $\mathbb{Z}^{2n}$ corresponds to the B_n-action on a disk containing punctures $P_1, \dots, P_n$. The key observation is that this disk is *filled* by the lamination encoded by the vector $(0, 1, \dots, 0, 1)$ (i.e. cutting this disk along its intersection with the lamination yields only disks and once-punctured disks).

3. Coordinates on virtual braid groups

Let us define an action by elements of VB_n on $\mathbb{Z}^{2n}$. Consider the actions on $\mathbb{Z}^4$ by σ and σ^{-1} as defined in (2.1) and (2.2), and define the action by ρ as the following permutation of coordinates:

$$(a, b, c, d) \cdot \rho = (c, d, a, b). \tag{3.1}$$

For a given vector $(a_1, b_1, \dots, a_n, b_n) \in \mathbb{Z}^{2n}$ we define the action by $\rho_i \in VB_n$, $i = 1, \dots, n - 1$:

$$(a_1, b_1, \dots, a_n, b_n) \cdot \rho_i = (a'_1, b'_1, \dots, a'_n, b'_n), \tag{3.2}$$

where $a'_k = a_k$, $b'_k = b_k$ for $k \neq i, i + 1$, and

$$(a'_i, b'_i, a'_{i+1}, b'_{i+1}) = (a_i, b_i, a_{i+1}, b_{i+1}) \cdot \rho. \tag{3.3}$$

The action by $\sigma_i^\varepsilon \in VB_n$ on $\mathbb{Z}^{2n}$ is defined according to (2.4) and (2.5).

Suppose that w is a word in the alphabet $\{\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}, \rho_1, \ldots, \rho_{n-1}\}$ representing an element of the group VB_n. Then we define

$$(a_1, b_1, \ldots, a_n, b_n) \cdot w = \begin{cases} ((a_1, b_1, \ldots, a_n, b_n) \cdot \rho_i) \cdot w', \text{ if } w = \rho_i w', \\ ((a_1, b_1, \ldots, a_n, b_n) \cdot \sigma_i^{\varepsilon}) \cdot w', \text{ if } w = \sigma_i^{\varepsilon} w'. \end{cases} \tag{3.4}$$

To show that the action by VB_n on $\mathbb{Z}^{2n}$ is correctly defined we will verify that the defining relations of the group VB_n are satisfied. Since ρ_i acts by permuting pairs of coordinates, the relations of the group S_n are obviously satisfied. The fact that relations of the group B_n are satisfied follows from [2]. So, we need to check only the case of mixed relations, i.e. that for any $v \in \mathbb{Z}^{2n}$ the relations

$$v \cdot (\sigma_i \, \rho_j) = v \cdot (\rho_j \, \sigma_i), \quad |i - j| > 1, \tag{3.5}$$

$$v \cdot (\rho_i \, \rho_{i+1} \, \sigma_i) = v \cdot (\sigma_{i+1} \, \rho_i \, \rho_{i+1}) \tag{3.6}$$

hold. Relations (3.5) hold obiously, because σ_i acts non-trivial only on the subvector $(a_i, b_i, a_{i+1}, b_{i+1})$ and ρ_j acts non-trivially only on the subvector $(a_j, b_j, a_{j+1}, b_{j+1})$. In order to verify (3.6) it is enough to consider the case $i = 1$ in the group VB_3. Denote

$$(x, y, z, t) \cdot \sigma = (a'(x, y, z, t), b'(x, y, z, t), c'(x, y, z, t), d'(x, y, z, t))$$

From

$$\begin{aligned} (a_1, b_1, a_2, b_2, a_3, b_3) \cdot (\rho_1 \, \rho_2 \, \sigma_1) &= (a_2, b_2, a_1, b_1, a_3, b_3) \cdot (\rho_2 \, \sigma_1) \\ = (a_2, b_2, a_3, b_3, a_1, b_1) \cdot \sigma_1 &= ((a_2, b_2, a_3, b_3) \cdot \sigma, a_1, b_1) \end{aligned}$$

and

$$\begin{aligned} &(a_1, b_1, a_2, b_2, a_3, b_3) \cdot (\sigma_2 \, \rho_1 \, \rho_2) = (a_1, b_1, (a_2, b_2, a_3, b_3) \cdot \sigma) \cdot \rho_1 \, \rho_2 \\ &= (a'(a_2, b_2, a_3, b_3), b'(a_2, b_2, a_3, b_3), a_1, b_1, c'(a_2, b_2, a_3, b_3), d'(a_2, b_2, a_3, b_3)) \cdot \rho_2 \\ &= (a'(a_2, b_2, a_3, b_3), b'(a_2, b_2, a_3, b_3), c'(a_2, b_2, a_3, b_3), d'(a_2, b_2, a_3, b_3), a_1, b_1) \\ &= ((a_2, b_2, a_3, b_3) \cdot \sigma, a_1, b_1) \end{aligned}$$

we see that (3.6) holds.

Example 3.1. Actions by some elements of VB_2 on $(0, 1, 0, 1) \in \mathbb{Z}^4$ are as follows:

$$\begin{aligned} &(0, 1, 0, 1) \cdot \sigma_1 \rho_1 = (0, 2, 1, 0), \\ &(0, 1, 0, 1) \cdot \sigma_1 \rho_1 \sigma_1 = (3, 0, 0, 2), \\ &(0, 1, 0, 1) \cdot \sigma_1 \rho_1 \sigma_1^{-1} = (-2, -1, 1, 3). \end{aligned}$$

Let us demonstrate that the forbidden relations are not satisfied. More exactly, we show that for $v = (0, 1, 0, 1, 0, 1)$ we get

$$v \cdot (\rho_1 \, \sigma_2 \, \sigma_1) \neq v \cdot (\sigma_2 \, \sigma_1 \, \rho_2), \tag{3.7}$$

$$v \cdot (\rho_2 \, \sigma_1 \, \sigma_2) \neq v \cdot (\sigma_1 \, \sigma_2 \, \rho_1). \tag{3.8}$$

Indeed, (3.7) holds because

$$(0,1,0,1,0,1)\cdot(\rho_1\,\sigma_2\,\sigma_1) = (0,1,0,1,0,1)\cdot\sigma_2\,\sigma_1 = (0,1,1,0,0,2)\cdot\sigma_1 = (2,0,0,1,0,2),$$

but

$$(0,1,0,1,0,1)\cdot(\sigma_2\,\sigma_1\,\rho_2) = (2,0,0,1,0,2)\cdot\rho_2 = (2,0,0,2,0,1).$$

Analogously, (3.8) holds because

$$(0,1,0,1,0,1)\cdot(\rho_2\,\sigma_1\,\sigma_2) = (0,1,0,1,0,1)\cdot\sigma_1\,\sigma_2 = (1,0,0,2,0,1)\cdot\sigma_2 = (1,0,2,0,0,3),$$

but

$$(0,1,0,1,0,1)\cdot(\sigma_1\,\sigma_2\,\rho_1) = (1,0,2,0,0,3)\cdot\rho_1 = (2,0,1,0,0,3).$$

Question 3.2. Is there any relation between our coordinates on VB_n and the invariant of virtual braids defined by Manturov in [5]?

4. Faithfulness of the VB_n-action on $\mathbb{Z}^{2n}$

In this section we will be concerned with the following

Question 4.1. Is the VB_n-action on $\mathbb{Z}^{2n}$ faithful? In other words, is it true that only the trivial element of VB_n acts as the identity on $\mathbb{Z}^{2n}$?

In computer experiments, we have tested several billion (10^9) random virtual braids with 3, 4 and 5 strands, but failed to find one that would provide a negative answer. The programs used for these tests (written in Scilab) can be obtained from B. Wiest's web page [7].

It should be stressed that nontrivial elements of VB_n may very well act trivially on individual vectors. Let us, for instance, look at the VB_3-action on $(0,1,0,1,0,1)$. The actions of ρ_1 and ρ_2 obviously fix this vector, but those of many other braids do, too. Here is one particularly striking example:

Example 4.2. The virtual 3 strand braid

$$\beta = \sigma_1\rho_2\sigma_1\sigma_2^{-1}\sigma_1\sigma_2\sigma_1^{-1}\rho_1\sigma_2\rho_1\sigma_1\rho_2\sigma_1^{-1}\rho_2\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1^{-1}\rho_2\sigma_1^{-1}$$

acts trivially on the vector $(0,1,0,1,0,1)$, and indeed in computer experiments, we found that it acted nontrivially on only about 0.25% of randomly generated vectors of $\mathbb{Z}^6$ with integer coefficients between -100 and 100. In this sense, β is "nearly a negative answer" to Question 4.1. Another one is the virtual braid $\sigma_2^{-1}\sigma_1\rho_2\sigma_2\sigma_1\sigma_2^{-1}\rho_2\sigma_1\rho_2\sigma_2\rho_1\sigma_2^{-1}\rho_1\sigma_1^{-1}\sigma_2^{-1}\rho_2\sigma_1^{-1}\sigma_2$ which also moves only about 0.6% of random vectors.

We now turn our attention to the case $n = 2$.

Example 4.3. It is known [5] that $\beta = (\sigma_1^2\rho_1\sigma_1^{-1}\rho_1\sigma_1^{-1}\rho_1)^2 \in VB_2$ and the identity element $\mathbf{e} \in VB_2$ cannot be distinguished by the Burau representation. Considering

actions by these elements on $(0,1,0,1)$ we get $(0,1,0,1)\cdot\beta = (85, 49, -90, -47)$, while $(0,1,0,1)\cdot\mathbf{e} = (0,1,0,1)$. Thus, these elements are distinguished by Dynnikov coordinates.

In fact, in the case $n = 2$ we have a positive answer to Question 4.1.

Theorem 4.4. *The VB_2–action on $\mathbb{Z}^4$ given by the above formulae is faithful.*

Proof. We will, in fact show a stronger result than Theorem 4.4, namely that the only element of VB_2 which acts trivially on a vector $(0, x, 0, y)$, where x and y are different positive integers, is the trivial one. As a simplest vector of this type one can take $(0, 2, 0, 1)$.

Consider the set of symbols $S = \{0, +, -, +0, -0\}$. With $s \in S$ we associate the following subset of $\mathbb{Z}$:

$$\begin{gathered} s = 0 \longleftrightarrow \{x \in \mathbb{Z} \,:\, x = 0\}, \\ s = + \longleftrightarrow \{x \in \mathbb{Z} \,:\, x > 0\}, \quad s = - \longleftrightarrow \{x \in \mathbb{Z} \,:\, x < 0\}, \\ s = +0 \longleftrightarrow \{x \in \mathbb{Z} \,:\, x \geqslant 0\}, \quad s = -0 \longleftrightarrow \{x \in \mathbb{Z} \,:\, x \leqslant 0\}. \end{gathered}$$

A quadruple (s_1, s_2, s_3, s_4), where $s_i \in S$, indicates the set all quadruples $(a, b, c, d) \in \mathbb{Z}^4$ such that a, b, c, d belongs to subset of $\mathbb{Z}$ associated with symbols s_1, s_2, s_3, s_4, respectively. For example,

$$(+, +0, 0, -) = \{(a, b, c, d) \in \mathbb{Z}^4 \,:\, a > 0,\, b \geqslant 0,\, c = 0,\, d < 0\}.$$

For instance, the vector $(-1, -1, 0, 4)$ belongs to sets associated with $(-, -, 0, +)$ and $(-, -0, +0, +)$, as well as with some other quadruples of symbols.

Let $\sigma, \sigma^{-1}, \rho : \mathbb{Z}^4 \to \mathbb{Z}^4$ be the transformations defined by (2.1), (2.2), (3.1). Let us apply a sequence of such transformations (without fragments $\sigma\sigma^{-1}$, or $\sigma^{-1}\sigma$, or $\rho\rho$) to the initial vector, say $(0, 2, 0, 1)$. The statement of the theorem will follow from the fact that we trace out a path in the diagram shown in Figure 1, where an arrow with labels β and $i.\beta$, where $i \in \{1, 2, \ldots, 14\}$ and $\beta \in \{\sigma, \sigma^{-1}, \rho\}$ from box (s_1, s_2, s_3, s_4) to box (t_1, t_2, t_3, t_4) means that the image of any element of the subset of $\mathbb{Z}^4$ associated with (s_1, s_2, s_3, s_4) under tranformation β belongs to the subset of $\mathbb{Z}^4$ associated with (t_1, t_2, t_3, t_4). The numbering of arrows by $1, 2, \ldots, 14$ is done for the reader's convenience in the further discussion of cases.

It is easy to see from (2.1), (2.2) and (3.1) that for any $\beta \in VB_2$ there is the following invariant: if $(a^*, b^*, c^*, d^*) = (a, b, c, d)\cdot\beta$ then $b^* + d^* = b + d$. In particular, if the initial vector is taken to be $(0, 2, 0, 1)$ then for any vector in the diagram the sum of its second and fourth coordinates is equal to 3.

Below to simplify expressions we will use the following notations:

$$(a', b', c', d') = (a, b, c, d)\cdot\sigma, \qquad (a'', b'', c'', d'') = (a, b, c, d)\cdot\sigma^{-1}.$$

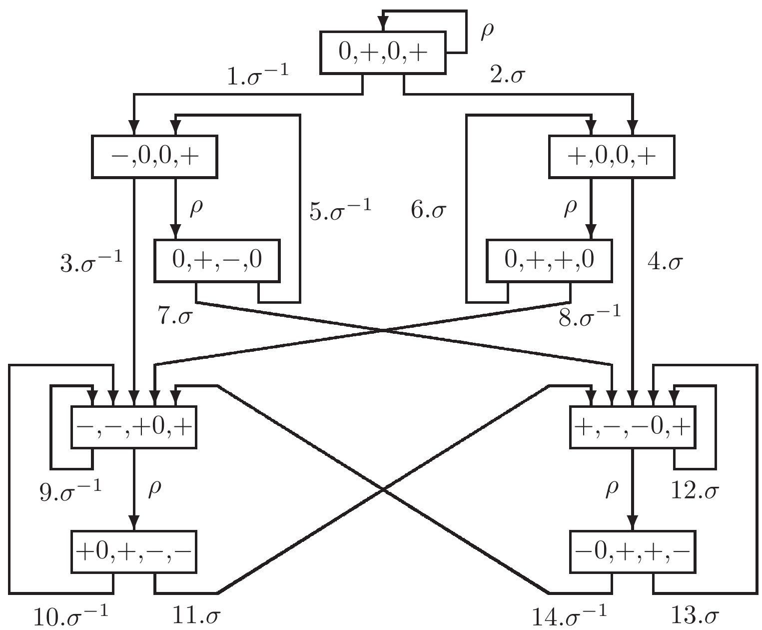

Fig. 1. The diagram of actions of VB_2.

By (2.1) and (2.2) we have

$$\begin{cases} a' = a + b^+ + (-a + b^- + c)^+, \\ b' = d - (a - b^- - c + d^+)^+, \\ c' = c + d^- + (a - c + d^+)^-, \\ d' = b + (a - b^- - c + d^+)^+, \end{cases} \qquad \begin{cases} a'' = a - b^+ - (a + b^- - c)^+, \\ b'' = d + (a + b^- - c - d^+)^-, \\ c'' = c - d^- - (-a + c + d^+)^-, \\ d'' = b - (a + b^- - c - d^+)^-, \end{cases} \tag{4.1}$$

First of all we remark that the arrows of the diagram related to the action by ρ hold obviously. The other actions will be considered case by case according to the numbering of the arrows.

<u>1.</u> Consider the action by σ^{-1} on $(0,+,0,+)$. Each element of $(0,+,0,+)$ has a form $(0,b,0,d)$ for some $b,d>0$. Since its image $(0,b,0,d)\cdot\sigma^{-1} = (-b,0,0,b+d)$ belongs to $(-,0,0,+)$, the corresponding arrow of the diagram is proven.

<u>2.</u> Consider the action by σ on $(0,+,0,+)$. Each element of $(0,+,0,+)$ has a form $(0,b,0,d)$ for some $b,d>0$. Since its image $(0,b,0,d)\cdot\sigma = (b,0,0,b+d)$ belongs to $(+,0,0,+)$, the corresponding arrow of the diagram is proven.

<u>3.</u> Consider the action by σ^{-1} on $(-,0,0,+)$. Let $a<0$, $d>0$ then $(a,0,0,d)\cdot\sigma^{-1} = (a, d+(a-d)^-, 0, -(a-d)^-) = (a,a,0,-a+d) \in (-,-,0,+) \subset (-,-,+0,+)$.

<u>4.</u> Consider the action by σ on $(+,0,0,+)$. Let $a,d>0$ then $(a,0,0,d)\cdot\sigma = (a,-a,0,a+d) \in (+,-,0,+) \subset (+,-,-0,+)$.

5. Consider the action by σ^{-1} on $(0,+,-,0)$. Let $b>0$, $c<0$ then $(0,b,c,0)\cdot\sigma^{-1}=(-b-(-c)^+,0,c-(c)^-,b-(-c)^-)=(-b+c,0,0,b)\in(-,0,0,+)$.

6. Consider the action by σ on $(0,+,+,0)$. Let $b,c>0$ then $(0,b,c,0)\cdot\sigma=(b+c,0,0,b)\in(+,0,0,+)$.

7. Consider the action by σ on $(0,+,-,0)$. Let $b>0$, $c<0$ then $(0,b,c,0)\cdot\sigma=(b,c,c,b-c)\in(+,-,-,+)\subset(+,-,-0,+)$.

8. Consider the action by σ^{-1} on $(0,+,+,0)$. Let $b,c>0$ then $(0,b,c,0)\cdot\sigma^{-1}=(-b,-c,c,b+c)\in(-,-,+,+)\subset(-,-,+0,+)$.

9. Let us demonstrate that $(-,-,+0,+)\cdot\sigma^{-1}\in(-,-,+0,+)$. Since $a<0$, $b<0$, $c\geqslant 0$, $d>0$, the action is given by formulae

$$\begin{cases} a''=a-(a+b-c)^+,\\ b''=d+(a+b-c-d)^-,\\ c''=c-(-a+c+d)^-,\\ d''=b-(a+b-c-d)^-. \end{cases}$$

Moreover, $a+b-c<0$ and $a+b-c-d<0$ imply that $(a,b,c,d)\cdot\sigma^{-1}=(a,a+b-c,c,-a+c+d)\in(-,-,+0,+)$.

10. Let us demonstrate that $(+0,+,-,-)\cdot\sigma^{-1}\in(-,-,+0,+)$. Since $a\geqslant 0$, $b>0$, $c<0$, $d<0$, the action is given by formulae

$$\begin{cases} a''=a-b-(a-c)^+,\\ b''=d+(a-c)^-,\\ c''=c-d-(-a+c)^-,\\ d''=b-(a-c)^-. \end{cases}$$

Moreover, $a-c>0$ implies that $(a,b,c,d)\cdot\sigma^{-1}=(-b+c,d,a-d,b)\in(-,-,+,+)\subset(-,-,+0,+)$.

11. Let us demonstrate that $(+0,+,-,-)\cdot\sigma\in(+,-,-0,+)$. Since $a\geqslant 0$, $b>0$, $c<0$, $d<0$, the action is given by formulae

$$\begin{cases} a'=a+b+(-a+c)^+,\\ b'=d-(a-c)^+,\\ c'=c+d+(a-c)^-,\\ d'=b+(a-c)^+. \end{cases}$$

Moreover, $a-c>0$ implies that $(a,b,c,d)\cdot\sigma=(a+b,d-a+c,c+d,b+a-c)\in(+,-,-,+)\subset(+,-,-0,+)$.

12. Let us demonstrate that $(+,-,-0,+)\cdot\sigma = (+,-,-0,+)$. Since $a > 0$, $b < 0$, $c \leqslant 0$, $d > 0$, the action is given by formulae

$$\begin{cases} a' = a + (-a+b+c)^+, \\ b' = d - (a-b-c+d)^+, \\ c' = c + (a-c+d)^-, \\ d' = b + (a-b-c+d)^+. \end{cases}$$

Moreover, $-a+b+c < 0$, $a-b-c+d > 0$, and $a-c+d > 0$ imply that $(a,b,c,d)\cdot\sigma = (a, -a+b+c, c, a-c+d) \in (+,-,-0,+)$.

13. Let us demonstrate that $(-0,+,+,-)\cdot\sigma \in (+,-,-0,+)$. Since $a \leqslant 0$, $b > 0$, $c > 0$, $d < 0$, the action is given by formulae

$$\begin{cases} a' = a + b + (-a+c)^+, \\ b' = d - (a-c)^+, \\ c' = c + d + (a-c)^-, \\ d' = b + (a-c)^+. \end{cases}$$

Moreover, $a-c < 0$ implies that $(a,b,c,d)\cdot\sigma = (b+c, d, d+a, b) \in (+,-,-,+) \subset (+,-,-0,+)$.

14. Let us demonstrate that $(-0,+,+,-)\cdot\sigma^{-1} = (-,-,+0,+)$. Since $a \leqslant 0$, $b > 0$, $c > 0$, $d < 0$, the action is given by

$$\begin{cases} a'' = a - b - (a-c)^+, \\ b'' = d + (a-c)^-, \\ c'' = c - d - (-a+c)^-, \\ d'' = b - (a-c)^-. \end{cases} \tag{4.2}$$

Moreover, $a-c < 0$ implies that $(a,b,c,d)\cdot\sigma^{-1} = (a-b, d+a-c, c-d, b-a+c) \in (-,-,+,+) \subset (-,-,+0,+)$. The proof is completed. $\square$

Remark 4.5. Theorem 4.4 allows us to introduce on VB_2 various "coordinate systems". For example, taking $(0,2,0,1)$ as the initial vector, for any $\beta \in VB_2$ one can define $(0,2,0,1)\cdot\beta$ as its coordinates. In this sense Theorem 4.4 gives an anolog of Dynnikov coordinates originally defined for braid groups.

Remark 4.6. It is shown in the proof of Theorem 4.4 that the VB_2-action is faithful on any vector of the form $(0,x,0,y)$, where x and y are different positive integers. Note that the action on some other vectors of $\mathbb{Z}^4$ can fail to be faithful. For example, $(0,0,0,1)\cdot\sigma = (0,0,0,1)$ and $(0,1,0,1)\cdot\rho = (0,1,0,1)$.

Remark 4.7. Let us define the norm of a quadruple $||(a,b,c,d)|| = |a|+|b|+|c|+|d|$. Obviously, the norm is invariant under the ρ-action. One can easily see from the proof of Theorem 4.4 that all the arrows labelled σ or σ^{-1} in Figure 1 increase the norm. For example, considering case 13, one gets $||(a,b,c,d)\cdot\sigma|| = ||(b+c,d,d+$

$a, b)|| = |b+c| + |d| + |d+a| + |b| > |a| + |b| + |c| + |d|$, because $a \leqslant 0$, $b > 0$, $c > 0$, $d < 0$. However, this property doesn't hold if one takes an arbitrary vector from $\mathbb{Z}^4$: $||(7,4,1,1)|| = 13$, but $||(7,4,1,1) \cdot \sigma|| = ||(3,1,6,1)|| = 11$.

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References

[1] V. Bardakov, The virtual and universal braids, *Fundamenta Mathematicae* **184** (2004), 1-18.

[2] P. Dehornoy, with I. Dynnikov, D. Rolfsen, B. Wiest, Ordering braids, *Math. Surveys and Monographs* **148**, AMS (2008)

[3] R. Fenn, R. Rimányi, C. Rourke, The braid-permutation group, *Topology* **36(1)** (1997), 123–135.

[4] L. Kauffman, Virtual knot theory, *Eur. J. Comb.* **20(7)** (1999), 669–690.

[5] V. O. Manturov, Recognition of virtual braids, *J. of Math. Sciences* **131(1)** (2005), 267–286.

[6] V. Vershinin, Generalization of braids and homologies, *J. Math. Sciences* **134(2)** (2006), 2002–2016.

[7] B. Wiest, `http://perso.univ-rennes1.fr/bertold.wiest`